

Parametric Oscillators

So far, have explored oscillators with varying frequencies, i.e.:

$$\ddot{x} + \omega^2(t) x = 0$$

For convenience:

$$\omega^2(t) = \omega_0^2 (1 + h \cos \Omega t) \quad , \quad h < 1$$

→ $\Omega \ll \omega_0 \Rightarrow$ adiabatic limit

⇒ adiabatic invariant theory.

→ $\Omega \gg \omega_0 \Rightarrow$ ponderomotive limit

⇒ ponderomotive force, quiver velocity, etc.

Now:

→ $\Omega \sim \omega_0$, especially $\Omega \sim 2\omega_0$

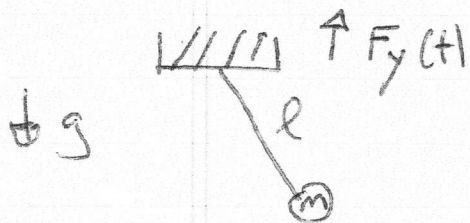
⇒ parametric instability.

Plan:

- i) ideas.
- ii) theory of Mathieu equation.
- iii) method of variation of parameters.

i) Parametric Instability - Ideas

→ consider pendulum with support acted on by vertical force



so $g \Rightarrow g - F_y(t)/m$

$\downarrow + \rightarrow \text{down}$

$$\therefore \ddot{\theta} = \ddot{\theta} + \frac{g}{l} \theta \rightarrow \ddot{\theta} + \left(\frac{g}{l} - \frac{a(t)}{l} \right) \theta = 0$$

let $a(t) = a_0 \cos(\alpha t)$

$$\Rightarrow \ddot{\theta} + \omega_0^2 \theta - \frac{a_0 \cos(\alpha t)}{l} \theta = 0$$

see small ϕ limit, No. 12

$\alpha \gg \omega_0$

of Mathieu's equation genre, i.e.

$$\ddot{x} + \omega_0^2 (1 + a \cos(\gamma t)) x = 0$$

$\omega^2 = \omega^2(t)$, hence parametric oscillator

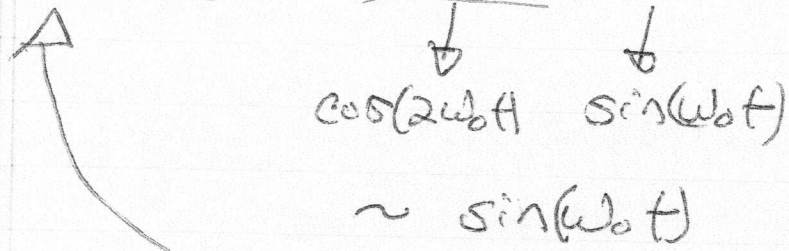
Parametric oscillator $\leftrightarrow \omega^2(t)$ periodic oscillation of effective frequency.

→ Some observations:

2) informal - consider what might happen?

for instability, observe can produce secularly if $\gamma \sim 2\omega_0$ via beat at fundamental

$$\ddot{x} + \omega_0^2 x + a \cos(\gamma t) \omega_0^2 x = 0$$



resonant drive of fundamental oscillator \Rightarrow secularly \rightarrow instability (why?)

• Solution of oscillator at ω_0 beats with parameter oscillation \Rightarrow secularly

• Parametric resonance at/near $\gamma \sim 2\omega_0$ (twice fundamental)

Note: here $\omega^2 = \omega^2(t) \Rightarrow \frac{\partial L}{\partial t} \neq 0$ energy not conserved

\Rightarrow work done on system (e.g. LGM oscillating pendulum support)

\leftrightarrow source of energy for instability

What is relation of this to 3-mode parametric instability calculation (2004)

Familiar example

- free asymmetric top: I_1, I_2, I_3
s/t

$$I_1 < I_2 < I_3$$

- Consider Poisson construction

i.e. L/L Fig: 51 (pg 117)

$$L_1^2 + L_2^2 + L_3^2 = L^2 = \text{const} \quad (\text{sphere})$$

$$\frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} = E = \text{const} \quad (\text{ellipsoid})$$

$$\frac{d\underline{L}}{dt} = -\underline{\Omega} \times \underline{L}, \quad \frac{d\underline{L}}{dt} + \underline{\Omega} \times \underline{L} = \underline{\gamma}$$

\Rightarrow picture

\Rightarrow linear theory: i.o. axes 1,3 \rightarrow stable
axes 2 \rightarrow unstable.

\sim parameters

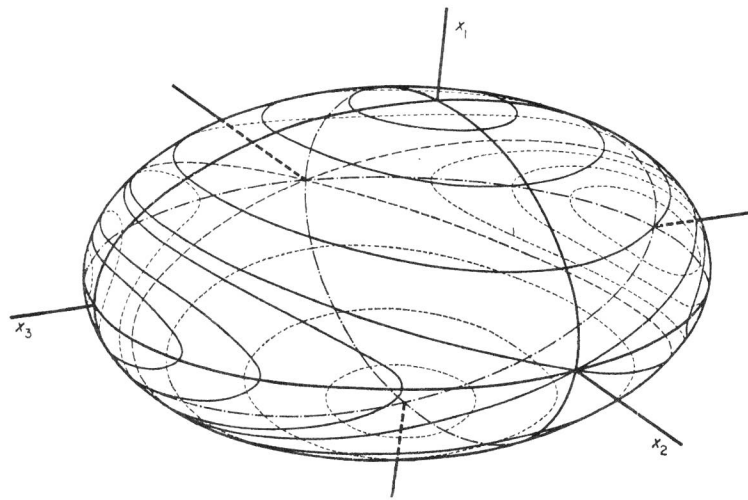


FIG. 51

(ii) Formal (Floquet theory) $\left\{ \begin{array}{l} \text{What} \\ \text{Mathematics} \\ \text{Predicts} \end{array} \right.$
 \Rightarrow (What type solution possible)
 - $\omega(t)$ periodic, with period $T = 2\pi/\gamma$

$\therefore \begin{cases} \omega(t+T) = \omega(t) \\ \text{eqn. invariant under } t \rightarrow t+T \end{cases}$

\therefore if $x_1(t), x_2(t)$ are 2 independent solutions of basic eqn.

$\Rightarrow x_1(t), x_2(t)$ must transform to linear combinations of themselves upon $t \rightarrow t+T$ (linear eqn)

and

can choose x_1, x_2 s.t

diagonalize transform.

$$\begin{cases} x_1(t+T) = \mu_1 x_1(t) \\ x_2(t+T) = \mu_2 x_2(t) \end{cases}$$

(here "can choose" means can diagonalize transformation matrix)

\rightarrow most general functions having this property are:

$$\begin{cases} x_1(t) = \mu_1^{t/T} \pi_1(t) \\ x_2(t) = \mu_2^{t/T} \pi_2(t) \end{cases}$$

where:

$$\begin{cases} \pi_1(t+T) = \pi_1(t) \\ \pi_2(t+T) = \pi_2(t) \end{cases}$$

- second, observe since linear equation
 \Rightarrow Wronskian constant.

$$\dot{x}_2 x_1 - \dot{x}_1 x_2 = \text{const.}$$

to see:

$$\begin{aligned} x_2 (\ddot{x}_1 + \omega^2(t) x_1) &= 0 & \Rightarrow \frac{d}{dt} (x_2 \dot{x}_1 - \dot{x}_2 x_1) &= 0 \\ x_1 (\ddot{x}_2 + \omega^2(t) x_2) &= 0 \end{aligned}$$

but

$$W(x_1, x_2) = (u_1, u_2)^{-1} W(x_1(t+T), x_2(t+T))$$

c.e. consider time translation by T

$$\rightarrow \boxed{u_1, u_2 = 1} \quad \left| \quad W(x_1, x_2) = \begin{pmatrix} u_2 & \frac{\partial T}{\partial t} \\ u_1 & \frac{\partial T}{\partial t} \end{pmatrix} \begin{pmatrix} u_1 & \frac{\partial T}{\partial t} \\ u_2 & \frac{\partial T}{\partial t} \end{pmatrix} - \begin{pmatrix} u_1 & \frac{\partial T}{\partial t} \\ u_2 & \frac{\partial T}{\partial t} \end{pmatrix} \begin{pmatrix} u_1 & \frac{\partial T}{\partial t} \\ u_2 & \frac{\partial T}{\partial t} \end{pmatrix} \right.$$

- Can also observe:

1) wells in oscillator regl, so
 $x(t)$ on ~~oscillator~~ $\rightarrow x^*$ a solution
 \Rightarrow solution

2) u_1, u_2 same as u_1^*, u_2^*
c.e.

$$\begin{cases} u_1 = u_2^* \\ u_2 = u_1^* \end{cases} \quad \text{I}$$

$$\text{or } \begin{cases} u_1 = u_1^* \\ u_2 = u_2^* \end{cases} \quad \text{both real} \quad \text{II}$$

if I, $u_1, u_2 = 1 \Rightarrow u_1 = 1/u_1^* \Rightarrow |u_1|^2 = |u_2|^2 = 1$
 $u_2 = 1/u_2^* \Rightarrow$
(trivial)

non-trivial case

9.

if \textcircled{II} $\mu_1, \mu_2 = 1$; μ_1, μ_2 real \Rightarrow

$$\Rightarrow \left[x_1(t) = \mu^{t/T} \pi_1(t), \quad x_2(t) = \mu^{-t/T} \pi_2(t) \right]$$

i.e. $\left. \begin{array}{l} \uparrow \text{ increasing} \\ \downarrow \text{ decreasing} \end{array} \right\} \text{ solution} \Rightarrow \left[\begin{array}{l} \text{parametric} \\ \text{instability} \end{array} \right]$

[N.B. Exponential, not secular growth]!
 \Rightarrow "true" instability is possible

\rightarrow Some Calculation (as basic structure of the solution established).

Consider Mathieu's eqn:

$$\ddot{x} + \omega^2 \left[1 + \underset{\substack{\downarrow \\ \text{strength } h}}{h} \cos \left[\underset{\substack{\rightarrow \\ \text{mismatch } \epsilon}}{(2\omega_0 + \epsilon)t} \right] \right] x = 0$$

bounds on ϵ for instability?

For solution, SHO \Rightarrow

$$x = a \cos(\omega_0 t) + b \sin(\omega_0 t)$$

so, in spirit of multiple-time-scale P.T.
(i.e. $\omega^2(t)$ enters via $h \ll 1 \Rightarrow$ expect slow time scale variation of coefficients)
mismatch USA \rightarrow key competition

$$x = a(t) \cos[(\omega_0 + \epsilon/2)t] + b(t) \sin[(\omega_0 + \epsilon/2)t]$$

\downarrow
coeffs become slowly varying

For More on Multiple time scale perturbation theory \rightarrow see Bender and Orszag, last chapter.

so, plugging in:

$$\begin{cases} \ddot{x} + \omega_0^2 [1 + h \cos[(2\omega_0 + \epsilon)t]] x = 0 \\ x = a(t) \cos[(\omega_0 + \epsilon/2)t] + b(t) \sin[(\omega_0 + \epsilon/2)t] \end{cases}$$

$$\ddot{x} = \left[a(t) \cos[(\omega_0 + \epsilon/2)t] + b(t) \sin[(\omega_0 + \epsilon/2)t] \right]$$

$$= -(\omega_0 + \epsilon/2)^2 a(t) \cos [\]$$

$$- 2(\omega_0 + \epsilon/2) \dot{a}(t) \sin [\]$$

$$+ \ddot{a} \cos [\] \quad \text{slow}$$

$$- (\omega_0 + \epsilon/2)^2 b(t) \sin [\]$$

$$+ 2(\omega_0 + \epsilon/2) \dot{b}(t) \cos [\]$$

$$+ \ddot{b}(t) \sin [\] \quad \text{slow.}$$

$$[\] \equiv [(\omega_0 + \epsilon/2)t]$$

so, neglecting slow \ddot{a} , \ddot{b} :

⇒

$$\begin{aligned}
 & - (\omega_0^2 + \omega_0 \epsilon + \cancel{\epsilon^2/4}) a(t) \cos[\] \\
 & - 2 \dot{a}(t) (\omega_0 + \epsilon/2) \sin[\] \\
 & - (\omega_0^2 + \omega_0 \epsilon + \cancel{\epsilon^2/4}) b(t) \sin[\] \\
 & + 2 \dot{b}(t) (\omega_0 + \epsilon/2) \cos[\] \\
 & + \omega_0^2 [a(t) \cos[\] + b(t) \sin[\]] \\
 & + \omega_0^2 h \cos[(2\omega_0 + \epsilon)t] [a(t) \cos[\] \\
 & + b(t) \sin[\]] = 0
 \end{aligned}$$

can simplify by:

- cancelling ω_0^2 terms

- drop $O(\epsilon^2)$.

80

8

$$\begin{aligned}
 & -\omega_0 \epsilon (a(t) \cos [\] + b(t) \sin [\]) \\
 & - 2\dot{a} (\omega_0 + \epsilon/2) \sin [\] + 2\dot{b} (\omega_0 + \epsilon/2) \cos [\] \\
 & + \omega_0^2 h \cos ((2\omega_0 + \epsilon)t) [a(t) \cos [\] + b(t) \sin [\]] \\
 & = 0
 \end{aligned}$$

Further, observe in \uparrow ;

$$\begin{aligned}
 & \cos [(\omega_0 + \epsilon/2)t] \cos [(2\omega_0 + \epsilon)t] \\
 & = \frac{1}{2} \cos [(\omega_0 + \epsilon/2)t] + \frac{1}{2} \cos [3(\omega_0 + \epsilon/2)t]
 \end{aligned}$$

resonant

drives at $\sim \omega$

non-resonant
 \rightarrow drives ω_0 osc.
 at $3\omega_0$

\rightarrow expect h.o.
 in h effect

\rightarrow drop

\rightarrow Resonant interaction is interesting one here..

$$\begin{aligned} & \neq -\omega_0 \epsilon (a(t) \cos[\] + b(t) \sin[\]) \\ & - 2 \dot{a} (\omega_0 + \epsilon/2) \sin[\] + 2 \dot{b} (\omega_0 + \epsilon/2) \cos[\] \\ & + \frac{\omega_0^2 h}{2} [a(t) \cos[\] - b(t) \sin[\]] \\ & = 0 \end{aligned}$$

Regrouping coefficients $\cos[\]$, $\sin[\]$;

$$\begin{aligned} & \sin[\] (-2\omega_0 \dot{a} - b\omega_0 \epsilon - \frac{\omega_0^2 h}{2} b) \\ & + \cos[\] (2\dot{b}\omega_0 - a\epsilon\omega_0 + \frac{1}{2} h\omega_0^2 a) = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow & (2\omega_0) \dot{a} + (\omega_0 \epsilon) b + (\frac{\omega_0^2 h}{2}) b = 0 \\ & (2\omega_0) \dot{b} - (\omega_0 \epsilon) a + (\frac{\omega_0^2 h}{2}) a = 0 \end{aligned}$$

$$\Rightarrow \left\{ \begin{aligned} \dot{a} + (\epsilon/2) b + (\omega_0 h/4) b &= 0 \\ \dot{b} - (\epsilon/2) a + (\omega_0 h/4) a &= 0 \end{aligned} \right.$$

Basic system of Eqs for Amplitude Variation

$$a(t) = a_0 e^{st}$$

$$b(t) = b_0 e^{st}$$

exponentially
growing/damping solution

⇒

$$s a_0 + (\epsilon/2 + i\omega h/4) b_0 = 0$$

$$\left(-\frac{\epsilon}{2} + \frac{i\omega h}{4}\right) a_0 + s b_0 = 0$$

$$\therefore \left\{ s^2 = \frac{\omega^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4} \left(\frac{\omega^2 h^2}{4} - \epsilon^2 \right) \right.$$

⇒ Parametric instability criterion

Growth rate

Observe:

- instability for:

$$\epsilon^2 = (\gamma - 2\omega_0)^2 < \frac{\omega^2 h^2}{4}$$

$\omega_0 \rightarrow$ Fundamental

$\gamma \rightarrow$ Parametric
variation freq.

amplitude of
variation
 $h^2 > 4(\gamma - \omega_0)^2 / \omega^2$

↑
i.e. sufficiently
close to resonance
⇒ growth.

(instability)

for $(\gamma - 2\omega_0)^2 > \omega^2 h^2 / 4 \rightarrow$ oscillation ⇒ stable

- amplitude of $\omega_0^2(t)$ variation sets
proximity threshold.

more generally, can show when $n\gamma = 2\omega_0$ ^{integer}
 \Rightarrow parametric resonance. of course, higher $n \Rightarrow$ resonance region $\sim h^n$

- with friction, find threshold for instability:

c.e. $(\gamma - 2\alpha)^2 < \left[\left(\frac{1}{2} h \omega_0 \right)^2 - 4\alpha^2 \right]$

\uparrow
friction coeff

c.e. P.I. growth must be damped!
 \Rightarrow Friction raises required h .

- Pumping on swing:

\rightarrow "pumping" \rightarrow change of I

$$\ddot{\theta} + \frac{mgl\theta}{I(t)} = 0$$

$$I(t) = I_0 + \epsilon I_1(t)$$

$$\ddot{\theta} + \frac{g}{l}\theta + \frac{\epsilon g}{l} \frac{\Delta I(t)}{I} \theta = 0$$

$+ \alpha \dot{\theta}$

need pump twice per cycle

\rightarrow Parametric instability is prototypical example of method of multiple scales P.I.

Note: - oscillation frequency ω_a

- growth:

$$\delta^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4}$$

$$\downarrow$$
$$(\gamma - 2\omega)^2 / 4$$

Points: $\gamma \ll \omega_a$.